Homework #8 Solutions

Section 16.2

2. $y'' - 3y' - 10y = 2x - 3.$

First find y_c: Solve $y'' - 3y' - 10y = 0$

Start with: $r^2 - 3r - 10 = 0 \rightarrow (r-5)(r+2) = 0$

So $r = 5$ or $r = -2$.

$$y_c = c_1 e^{5x} + c_2 e^{-2x}$$

"Guess" $y_p = Ax + B$.

$$y_p'' - 3y' - 10y = -3A - 10(Ax + B) = 2x - 3.$$

So $-10A = 2 \longrightarrow A = -\frac{1}{5}$

$-3A - 10B = -3 \longrightarrow \frac{3}{5} - 10B = -3$

$-10B = -\frac{18}{5} \rightarrow B = \frac{9}{25}.$

So $\boxed{y = \frac{-x}{5} + \frac{9}{25} + c_1 e^{5x} + c_2 e^{-2x}}$

3. $y'' - y' = \sin x.$

First find y_c: Solve $y'' - y' = 0$

Start with $r^2 - r = 0 \rightarrow (r-1)r = 0$

So $y_c = c_1 + c_2 e^x$

"Guess" y_p: $A \sin x + B \cos x$. Then $y_p' = A\cos x - B \sin x$

$y_p'' = -A \sin x - B \cos x$

$$y_p'' - y_p' = -A\sin x - B\cos x + B\sin x - A\cos x = \sin x$$

$$\left.\begin{aligned} B - A &= 1 \\ A + B &= 0 \end{aligned}\right\} \quad 2B = 1 \rightarrow B = \tfrac{1}{2}$$

$A = -\frac{1}{2}.$

So $\boxed{y = \frac{-\sin x}{2} + \frac{\cos x}{2} + c_2 e^x + c_1}$

8. $y''+y = 2x+3e^x$

First find y_c: Solve $y''+y=0$

$$r^2+1=0 \rightarrow r=\pm i$$

So $y_c = c_1\sin x + c_2\cos x$

"Guess" y_p: $Ax+B+Ce^x$

So $y_p' = A+Ce^x$

$y_p'' = Ce^x$.

$y_p''+y_p = 2Ce^x + Ax + B = 2x + 3e^x$

So $A=2$, $2C=3 \rightarrow C=\frac{3}{2}$.

$B=0$

So $\boxed{y = 2x + \frac{3}{2}e^x + c_1\sin x + c_2\cos x}$

12. $y''+3y'+2y = e^{-x}+e^{-2x}-x$.

First find y_c: Solve $y''+3y'+2y=0$.

$$r^2+3r+2=0 \rightarrow (r+2)(r+1)=0$$

So $y_c = c_1e^{-2x}+c_2e^{-x}$.

"Guess" y_p: $Axe^{-x}+Bxe^{-2x}+Cx+D$.

$y_p' = -Axe^{-x}+Ae^{-x} + -2Bxe^{-2x}+Be^{-2x}+C$

$y_p'' = Axe^{-x}-Ae^{-x}-Ae^{-x}+4Bxe^{-2x}-2Be^{-2x}-2Be^{-2x}$

$= Axe^{-x}-2Ae^{-x}+4Bxe^{-2x}-4Be^{-2x}$

$$\begin{aligned} y_p''+3y'+2y = {} & \cancel{Axe^{-x}} - 2Ae^{-x} + \cancel{4Bxe^{-2x}} - 4Be^{-2x} \\ & \cancel{-3Axe^{-x}} + 3Ae^{-x} \cancel{-6Bxe^{-2x}} + 3Be^{-2x} + 3C \\ & \cancel{+2Axe^{-x}} + \cancel{2Bxe^{-2x}} + 2D + 2Cx \end{aligned}$$

$= Ae^{-x} - Be^{-2x} + 2Cx + 3C + 2D = e^{-x}+e^{-2x}-x$.

So $A=1$, $-B=1$, $2C=-1$, $3C+2D=0$.

$B=-1$ $\quad C=-\frac{1}{2}$ $\quad -\frac{3}{2}+2D=0 \rightarrow D=\frac{3}{4}$.

So $\boxed{y = xe^{-x} - xe^{-2x} - \frac{1}{2}x + \frac{3}{4} + c_1e^{-2x} + c_2e^{-x}}$

18. $y'' + y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$

First find y_c: Solve $y'' + y = 0$

$$r^2 + 1 = 0 \to r = \pm i, \text{ so } y_c = c_1 \sin x + c_2 \cos x.$$

So let $y_1 = \sin x, \quad y_2 = \cos x.$

Solve for v_1, v_2:

$$v_1' y_1 + v_2' y_2 = 0, \quad \rightsquigarrow \quad v_1' \sin x + v_2' \cos x = 0$$

$$v_1' y_1' + v_2' y_2' = \tan x \quad \rightsquigarrow \quad v_1' \cos x + v_2' (-\sin x) = \tan x.$$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x}{-\sin^2 x - \cos^2 x} = \frac{\sin x}{-1} = -\sin x$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{-1} = \frac{-\sin^2 x}{\cos x}$$

So $v_1 = \int -\sin x \, dx = \cos x$

$$v_2 = \int \frac{-\sin^2 x}{\cos x} dx = -\int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= -\int \sec x \, dx + \int \cos x \, dx$$

$$= -\ln |\sec x + \tan x| + \sin x.$$

So $y = \sin x \cos x + -\ln|\sec x + \tan x| \cdot \cos x + \sin x \cos x + c_1 \sin x + c_2 \cos x$

$$= \boxed{2 \sin x \cos x - \cos x \cdot \ln |\sec x + \tan x| + c_1 \sin x + c_2 \cos x}$$

19. $y'' + y = \sin x.$

From problem 18, $y_c = c_1 \sin x + c_2 \cos x.$

$v_1' \sin x + v_2' \cos x = 0$

$v_1' \cos x - v_2' \sin x = \sin x$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sin x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = -\sin x \cos x = -\frac{1}{2} \sin 2x.$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sin x \end{vmatrix}}{-1} = -\sin^2 x = \frac{\cos 2x - 1}{2}$$

19. cont.

So $v_1 = \int -\frac{1}{2}\sin 2x \, dx = \dfrac{\cos 2x}{4}$

$v_2 = \int \dfrac{\cos 2x - 1}{2} \, dx = \dfrac{\sin 2x}{4} - \dfrac{x}{2}$

Thus, $\boxed{y = \dfrac{\sin x \cos 2x}{4} + \dfrac{\cos x \sin 2x}{4} - \dfrac{x\cos x}{2} + C_1 \sin x + C_2 \cos x}$

28. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x \cos x, \quad x > 0.$

First find y_c: Solve $y'' - y' = 0$

$r^2 - r = 0 \rightarrow r(r-1) = 0.$

So $y_c = c_1 + c_2 e^x.$

Let $y_1 = 1, \quad y_2 = e^x.$

Solve for v_1, v_2:

$v_1' y_1 + v_2' y_2 = 0 \longrightarrow v_1' + e^x v_2' = 0$

$v_1' y_1' + v_2' y_2' = e^x \cos x \longrightarrow v_2' e^x = e^x \cos x$

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ e^x \cos x & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-e^{2x}\cos x}{e^x} = -e^x \cos x$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & e^x \cos x \end{vmatrix}}{e^x} = \cos x$$

So integrate to get

$v_1 = \int -e^x \cos x \, dx = -e^x \sin x + \int e^x \sin x \, dx$

$= -e^x \sin x - e^x \cos x + \int e^x \cos x \, dx$

$\Rightarrow v_1 = \dfrac{-e^x \sin x - e^x \cos x}{2}.$

$v_2 = \int \cos x \, dx = \sin x.$

Thus, $y = \dfrac{-e^x \sin x}{2} - \dfrac{e^x \cos x}{2} + (\sin x) e^x + c_1 + c_2 e^x$

$= \boxed{\dfrac{e^x \sin x}{2} - \dfrac{e^x \cos x}{2} + C_1 + C_2 e^x}$

29. $y'' - 5y' = xe^{5x}$, $\quad y_p = Ax^2e^{5x} + Bxe^{5x}$.

$$y_p' = 5Ax^2e^{5x} + 2Axe^{5x} + 5Bxe^{5x} + Be^{5x}$$

$$y_p'' = 25Ax^2e^{5x} + 10Axe^{5x} + 10Axe^{5x} + 2Ae^{5x} + 25Bxe^{5x} + 5Be^{5x} + 5Be^{5x}$$

$$= 25Ax^2e^{5x} + (20A + 25B)xe^{5x} + (2A + 10B)e^{5x}$$

$$y_p'' - 5y_p' = 25Ax^2e^{5x} + (20A+25B)xe^{5x} + (2A+10B)e^{5x}$$
$$\underline{- 25Ax^2e^{5x} - 10Axe^{5x} - 25Bxe^{5x} - 5Be^{5x}}$$

$$= 10Axe^{5x} + (2A + 5B)e^{5x} = xe^{5x}$$

So $10A = 1 \longrightarrow A = \frac{1}{10}$

$2A + 5B = 0 \longrightarrow \frac{1}{5} + 5B = 0 \rightarrow B = -\frac{1}{25}$

Thus, $\boxed{y_p = \frac{1}{10}x^2e^{5x} - \frac{1}{25}xe^{5x}}$

Now find y_c: Solve $y'' - 5y' = 0$

$r^2 - 5r = 0 \rightarrow r(r-5) = 0$.

So $y_c = c_1 + c_2e^{5x}$

Thus, $\boxed{y = c_1 + c_2e^{5x} + \frac{1}{10}x^2e^{5x} - \frac{1}{25}xe^{5x}}$

44. $y'' + 9y = 9x - \cos x$.

First find y_c: solve $y'' + 9y = 0$

$r^2 + 9 = 0 \rightarrow r = \pm 3i$.

So $y_c = c_1\sin 3x + c_2\cos 3x$.

Use undetermined coefficients: $y_p = Ax + B + C\sin x + D\cos x$.

$y_p' = A + C\cos x - D\sin x$

$y_p'' = -C\sin x - D\cos x$.

So $y_p'' + 9y = -C\sin x - D\cos x + 9Ax + 9B + 9C\sin x + 9D\cos x$

$= 9x - \cos x$

So $8C = 0$, $8D = -1$, $9A = 9$, $9B = 0$.

$D = -\frac{1}{8}$, $A = 1$.

So $\boxed{y = x - \frac{\cos x}{8} + c_1\sin 3x + c_2\cos 3x}$

Section 16.3

2. 8 lb weight stretches spring 4ft.
weight released 2ft above equilibrium position
w/ downward velocity 3 ft/s.

From the book, the equation we use is

$$my'' + \delta y' + ky = 0$$

$m = 8\text{lb}$, $\delta = 1.5$, k = spring constant.

To solve for k: $mg = ks$

$$8 \cdot 32 = k \cdot 4 \Rightarrow k = 64.$$

So we get

$$8y'' + 1.5y' + 64y = 0$$
$$y(0) = -2$$
$$y'(0) = 3$$